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Quantum rotors with regular frustration and the quantum Lifshitz point

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Abstract. We have discussed the zero-temperature quantum phase transition in n-component quantum rotor Hamiltonian in the presence of regular frustration in the interaction. The phase diagram consists of ferromagnetic, helical and quantum paramagnetic phase, where the ferro-para and the helical-para phase boundary meets at a multicritical point called a (d, m) quantum Lifshitz point where (d, m) indicates that the m of the d spatial dimensions incorporate frustration. We have studied the Hamiltonian in the vicinity of the quantum Lifshitz point in the spherical limit and also studied the renormalisation group flow behaviour using standard momentum space renormalisation technique (for finite n). In the spherical limit $(n \to \infty)$ one finds that the helical phase does not exist in the presence of any nonvanishing quantum fluctuation for $m = d$ though the quantum Lifshitz point exists for all $d > 1 + m/2$, and the upper critical dimensionality is given by $d_u = 3+m/2$. The scaling behaviour in the neighbourhood of a quantum Lifshitz point in d dimensions is consistent with the behaviour near the classical Lifshitz point in $(d+z)$ dimensions. The dynamical exponent of the quantum Hamiltonian z is unity in the case of anisotropic Lifshitz point $(d > m)$ whereas $z = 2$ in the case of isotropic Lifshitz point $(d = m)$. We have evaluated all the exponents using the renormalisation flow equations along-with the scaling relations near the quantum Lifshitz point. We have also obtained the exponents in the spherical limit $(n \to \infty)$. It has also been shown that the exponents in the spherical model are all related to those of the corresponding Gaussian model by Fisher renormalisation.

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1 Introduction

The study of phase transitions driven by quantum fluctuations in pure and frustrated spin systems have been an exciting area of recent research [1–6]. The Ising model in a transverse field [1] and its n-component generalization, the rotor models [2–6], are among the simplest systems to exhibit a zero-temperature quantum phase transition. In the absence of any quenched disorder the zero-temperature transitions in the afore-mentioned models (in d dimensions) have generally the same universality as the thermal phase transition in the equivalent classical model in $(d+1)$ dimensions [7] except at special points. In the presence of quenched disorder, the situation is fairly complicated since the disorder is frozen in time, and thus has no dynamical fluctuation.

The quantum rotor Hamiltonian is written as [2]

$$H_r = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{x}_i\mathbf{x}_j + \frac{1}{2In}\sum_i L_i^2, \quad \text{with} \quad \mathbf{x}_i^2 = n, \quad (1)$$

where $\mathbf{x}_i$ are n-component, fixed length vectors occupying the N sites of a d dimensional hypercubic lattice and the operator $L_i^2 = (1/2)\sum_{\mu\nu}(L_i^{\mu\nu})^2$ is the invariant formed from the asymetric rotor space angular momentum tensor $L_i^{\mu\nu} = x_i^\mu p_i^\nu - x_i^\nu p_i^\mu$. The moment of inertia term, which determines the strength of the quantum fluctuations, is denoted by I and J_{ij}'s denote the interaction among the rotors. In the zero-temperature limit, with the increase of the $1/I$, one eventually gets a transition from the ferromagnetic (symmetry broken) phase a quantum disordered (paramagnetic) ground state. The $n = 1$ limit of the above Hamiltonian corresponds to classical Ising-like spin models in the presence of a transverse field. Although this rotor-like systems (for $n > 1$) has got very limited scope of experimental investigations but as mentioned previously the model exhibits a nontrivial zero-temperature phase transition and thereby might be helpful in the studies of other systems showing quantum phase transitions [4].

The phase diagram and the critical behaviour associated with classical systems having regular frustration [8] in the interaction and the classical Lifshitz point [9,10] have been studied in details. The thermal phase boundary of a classical n-vector Hamiltonian consists of

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the ferromagnetic phase (with wave vector $q = 0$), the helically ordered phase (with nonzero ordering wave vector) and the paramagnetic phase. The ferro-para and helical-para phase boundary meets at a classical (d, m) Lifshitz point where (d, m) denotes that the interaction in the m of the d spatial directions of the lattice incorporate frustration [9]. More generally speaking, an m-fold Lifshitz point is characterised by an instability associated with the absence of quadratic terms of the form q_α^2 in the effective Landau-Ginzburg-Wilson Hamiltonian for all $\alpha = 1, 2, ...m \leq d$ [10]. Classically, it requires a pair of correlation length exponents (ν) and the Fisher exponents (η) to describe the transition at the Lifshitz point [9]. It has also been observed that in the classical case, for $m > d-1$, the helical long range order ceases to exist [10].

We have studied n-component quantum rotor Hamiltonian in the presence of regular frustration in the interaction. The zero-temperature saddle point (in the spherical or $n \to \infty$ limit) study of the phase diagram (Sect. 2) of the system clearly shows a ferro and modulated phase meeting at a point, we call a quantum (d, m) Lifshitz point. In the Section 3 we have employed a renormalisation group transformation to extract the information about the phase transition around such a quantum Lifshitz point with n finite. The renormalisation group studies and the scaling relations indicate that in case of the isotropic Lifshitz point, the dynamic exponent $z = 2$, in contrast to $z = 1$ in the anisotropic case [11]. We conclude that in the anisotropic case ($d > m$) the quantum Lifshitz point is essentially equivalent to a $(d + 1, m)$ classical Lifshitz point, as expected in absence of quenched disorder but the Isotropic case corresponds to a $d+2$ dimensional classical case with all the directions having frustrated interactions. In Section 4, we propose the modified hyperscaling relations and other scaling relations and in the Section 5, the exponents associated with the Lifshitz point are evaluated in the spherical limit ($n \to \infty$) evaluating the bubble diagrams and using "Fisher renormalisation" of the Gaussian exponents of the corresponding model.

2 The study of the phase diagram in the spherical limit

In this section, we shall study the quantum rotor Hamiltonian with regular frustration in the interaction J_{ij} (nearest neighbour interaction is ferromagnetic J_1, next nearest neighbour interaction being antiferromagnetic J_2) in the spherical limit ($n \to \infty$) and explore the phase diagram. Clearly, at zero temperature, the ground state is ferromagnetically ordered or spiral-like ordered depending upon the amount of frustration $\kappa = |J_2|/J_1$. For Axial next-nearest neighbour Ising (ANNNI) like systems the classical ground state is ferromagnetically ordered for $\kappa \leq 1/4$ and shows spiral structure for higher values of κ. Clearly, the phase diagram of the model consists of an isotropic Lifshitz point [9,10] where the ferro, spiral and para phases meet.

To evaluate the partition function of the quantum rotor Hamiltonian (1), we consider the path-integral representation. Using the fact that the angular momentum term in the Hamiltonian is the angular part of the n dimensional Laplacian and the length of the the rotors are fixed, one obtains a path integral representation of the partition function of the form [2,6]

$$Z = \int D\mathbf{x} \exp\left[\int_0^\beta d\tau \mathcal{L}(\mathbf{x}(\tau))\right], \tag{2a}$$

with the effective classical action given by

$$\mathcal{L}(\mathbf{x}(\tau)) = -\frac{1}{2} I \sum_i |\partial_\tau \mathbf{x}_i(\tau)|^2 + \frac{1}{2} \sum_{ij} J_{ij} \mathbf{x}_i \mathbf{x}_j,$$

with

$$\mathbf{x}_i^2(\tau) = n. \tag{2b}$$

Here $\mathbf{x}_i(\tau)$'s are classical imaginary time-dependent n-component vectors of length $\sqrt{n}$.

In the spherical limit, the saddle point condition at zero temperature ($T \to 0$) [12,13], namely $\partial f/\partial\lambda = 0$, where the analog of free energy $f(\lambda) = -\ln \int D\mathbf{x}[\int_0^\beta \mathcal{L}(\mathbf{x}, \lambda)]$, with λ as the Lagrange's undetermined multiplier, can be written as [14]

$$\sum_q \frac{1}{2I\omega_q} = 1, \tag{3a}$$

where

$$I\omega_q^2 = \lambda - J(q), \tag{3b}$$

and $J(q)$ is the Fourier transform of J_{ij}. Converting the sum over wavevector $\mathbf{q}$ to integral, condition (3a) becomes

$$\frac{1}{2\sqrt{I}} \int_0^{2\pi} \frac{d^d q}{(2\pi)^d} \frac{1}{(\lambda - J(q))^{1/2}} = 1, \tag{4}$$

where

$$\begin{aligned} J(q) =& (\cos q_1 + \cos q_2 + ... \cos q_d) \\ & - \kappa(\cos 2q_1 + \cos 2q_2 + ... + \cos 2q_m). \end{aligned} \tag{5a}$$

Defining $p = 1 - 4\kappa$ for future convenience, we get

$$J(q, p) = 2\sum_{i=1}^d \cos q_i - \frac{(1-p)}{4} \sum_{i=1}^m \cos 2q_i. \tag{5b}$$

For $p > 0$ the classical ground state is ferromagnetic (with zero modulation, all $q_\alpha = 0$) and for $p < 0$ corresponds to the helical phase (with $\cos q_\alpha = 1/(1-p)$, $\alpha = 1, 2..., m$). The case $p = 0$ is the special point in the phase diagram where the ferromagnetic and the helical phase meet, which, as we defined previously, corresponds in this case to the quantum Lifshitz point.

In order to look for a phase transition in the model, we are to check whether equation (4) has a solution for all values of I and p [12]. If the integral on the right hand side diverges for $\lambda \to J(\mathbf{q_0})$, where $\mathbf{q_0}$ is the ordering wavevector, then the saddle point equation has a solution for all values of I and p. The ground state energy and its derivatives are continuous functions of I and no phase transition occurs. If the integral converges, a saddle point exists above a certain critical value I_c of the parameter I which indicate a quantum phase transition (phase transition driven by the quantum fluctuations induced by the angular momentum term). So, the critical condition obtained from equation (5b) can be written as

$$2\sqrt{I_c} = \int_0^{2\pi} \frac{d^d q}{(2\pi)^d} \frac{1}{(\lambda_s - J(q))^{1/2}}, \tag{6a}$$

where

$$\lambda_s(p) = \frac{1}{4}(4d - m + mp) \quad \text{for} \quad p > 0 \tag{6b}$$

and

$$\lambda_s(p) = \frac{1}{4}(1-p)^{-1}\left[4d - m - (4d-2m)p + p^2\right] \quad \text{for } p < 0 \tag{6c}$$

To study the phase diagram in the vicinity of the quantum Lifshitz point ($p \to 0$), the ordering wave vector $q_\alpha \to (-p)^{1/2}$. One can simplify $J(q,p)$ using the form [10]

$$J(q,p) = J_0 - \frac{1}{c}\left[(q_\alpha^2 + \frac{p}{2})^2 + q_\beta^2\right] \tag{7}$$

where $q_\alpha^2 = \sum_{i=1}^m q_i^2$ and $q_\beta^2 = \sum_{i=m+1}^d q_i^2$ and c is some constant. With the above simplication of $J(q)$ we find using equation (6a), the critical condition at the quantum Lifshitz point ($p = 0$) as

$$2(\sqrt{I_c})_L = \int_0^{2\pi} \frac{d^d q}{(2\pi)^d} \frac{1}{(q_\alpha^4 + q_\beta^2)^{1/2}}, \qquad \sim \int d^m q_\alpha \frac{1}{(q_\alpha^4)^{(1-d+m)/2}}, \tag{8}$$

integrating over the q_β's ($d-m$ components of $\mathbf{q}$). Clearly, the above integral is convergent for any value of $d > 1 + m/2$. So we find a quantum Lifshitz point for the nonzero value of I only when $d > 1 + m/2$. The ferro-para phase boundary in the vicinity of a (d,m) is written as

$$2(\sqrt{I_c})_F = 2(\sqrt{I_c})_L + \int_0^{2\pi} \frac{d^d q}{(2\pi)^d}\left(\frac{1}{(pq_\alpha^2 + q_\alpha^4 + q_\beta^2)^{1/2}} - \frac{1}{(q_\alpha^4 + q_\beta^2)^{1/2}}\right), \tag{9}$$

and similarly the helical to para phase boundary is given as

$$2(\sqrt{I_c})_H = 2(\sqrt{I_c})_L + \int_0^{2\pi} \frac{d^d q}{(2\pi)^d}\left(\frac{1}{(\frac{p^2}{4} + pq_\alpha^2 + q_\alpha^4 + q_\beta^2)^{1/2}} - \frac{1}{(q_\alpha^4 + q_\beta^2)^{1/2}}\right). \tag{10}$$

It may be mentioned that the subscripts F, H and L in the equations (8) to (10) refer to the the critical values of I at the ferro-para, helical-para boundaries and at the Lifshitz point respectively.

For $1 + m/2 < d < 3 + m/2$, we have

$$2(\sqrt{I_c})_F = 2(\sqrt{I_c})_L + A_\pm |p|^{d-m/2-1} + Cp + O(p^2), \tag{11}$$

where the subscript + or − refers to the ferro-para transition ($p > 0$) and spiral-para transitions respectively. For $1 + m/2 < d < 2 + m/2$ the amplitude is given as [10]

$$A_+ = -1/2cK_m K_{d+1-m} \times B(d-1-m/2, 2-d+m/2)B[1/2d-1/2, 1/2(d+1-m)], \tag{11a}$$

and

$$A_- = -2^{d-m/2}\pi c K_m K_{d-m} \times \operatorname{cosec} \pi(d-m/2)\cos(1/2\pi(d+1))B(1/2m, d-m), \tag{11b}$$

where $K_d = 2^{-(d-1)}\pi^{-d/2}[\Gamma(1/2p)]^{-1}$, and $B(x,y)$ is the *beta*-function of x and y; c is a constant.

Clearly, for $d \to m$, the amplitude A_- vanishes, signalling the nonexistence of helical phase for any finite value of quantum fluctuations for $m = d$ in contrast to the classical case where $m = d - 1$ is marginal [10]. This result has also been obtained using the renormalisation group calculations with the one-loop approximation [11]. Here, the results for the crossover exponent and the universal amplitude ratio perfectly matches with the results obtained in the classical case with $d = d_{classical} + 1$. In the quantum case, the equation of the upper critical critical dimensionality line is modified as $d_u = 3 + m_u/2$. We show in the Figure 1. The schematic phase diagram of the phase boundary of a quantm rotor Hamiltonian in the neighbourhood of a quantum Lifshitz point.

3 Renormalisation group study for finite n

In this section, we shall study the exponents and the critical behaviour in the vicinity of a quantum Lifshitz point (with finite n) both in the isotropic ($m = d$) and anisotropic case ($m < d$), using the ϵ expansion around the upper critical dimension. Following the discussion of the last section, the quantum rotor partition function in the imaginary time representation, with relaxed spin constraint, can be put in the form [2,11]

$$\mathcal{Z} = \int D\mathbf{x}_i(\tau)\exp(-\mathcal{A}) \tag{12}$$

with the effective classical action given as

$$\mathcal{A} = \int_0^\beta d\tau \left(\mathcal{L}_0(\tau) - \sum_{ij} J_{ij}\mathbf{x}_i(\tau)\mathbf{x}_j(\tau)\right) \tag{13}$$

where

$$\mathcal{L}_0 = \frac{1}{2g}\sum_i(\partial_\tau \mathbf{x}_i)^2 + r\sum_i \mathbf{x}_i^2 + u\sum(\mathbf{x}_i^2)^2. \tag{14}$$

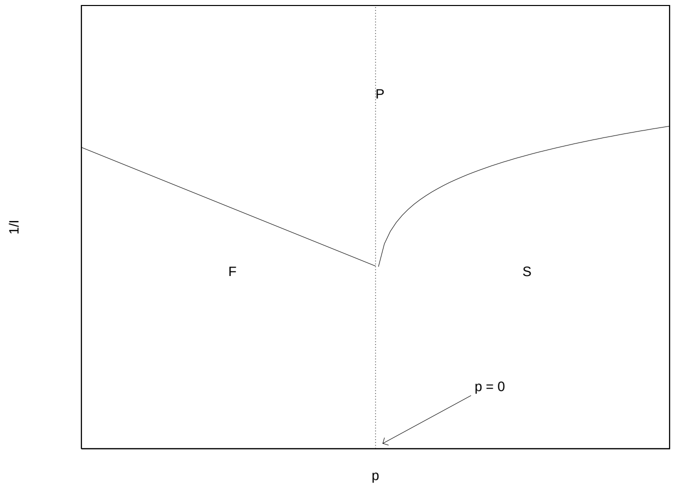

Fig. 1. The schematic phase diagram of the quantum rotor Hamiltonian is shown. $1/I$ denotes the strength of the quantum fluctuations and p is the measure of frustration. $p = 0$ point corresponds to the quantum Lifshitz point. F, S, P correspond to the ferromagnetic, spiral and paramagnetic phase respectively.

Here, the mass parameter r is the measure of the strength of the quantum fluctuations which in the case of $n = 1$ is proportional to the strength of the transverse field [7]. We recast the n-component rotor action in the Fourier space, with next nearest neighbour interaction in the m of the d spatial directions as

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\frac{d\omega}{2\pi} \\ &\times \left[\omega^2 + r + pq_\alpha^2 + q_\beta^2 + (q_\alpha^2)^2 + c_\gamma q_\alpha^2 q_\beta^2 + c_\delta (q_\beta^2)^2\right] \mathbf{x}_{\mathbf{q}}(i\omega) \\ &\times \mathbf{x}_{-\mathbf{q}}(-i\omega) + u\int \frac{d\omega_1}{2\pi}...\frac{d\omega_4}{2\pi}\int \frac{d^dq_1}{(2\pi)^d}... \\ &...\frac{d^dq_4}{(2\pi)^d}\delta^d(\mathbf{q}_1 + ... + \mathbf{q}_4)\delta(\omega_1 + ... + \omega_4) \\ &\times (\mathbf{x}_{\mathbf{q}_1}(i\omega_1)\cdot \mathbf{x}_{\mathbf{q}_2}(i\omega_2))(\mathbf{x}_{\mathbf{q}_3}(i\omega_3)\cdot \mathbf{x}_{\mathbf{q}_4}(i\omega_4)), \qquad (15)\end{aligned}$$

where $\mathbf{x}_{\mathbf{q}}(i\omega)$ are n-component Fourier space representations of $\mathbf{x}$ and ω's are continuous Matsubara frequencies and we have retained upto the quartic term $\mathbf{x}_{\mathbf{q}}$ in the effective action. Here the model is spherically symmetric in the subspace $\mathbf{q}_\alpha = (q_1, q_2, ..., q_m)$ and $\mathbf{q}_\beta = (q_{m+1}, ..., q_d)$. The equation (15) represents the generic action for a (d, m) quantum Lifshitz point. The parameter p arises in the present case due to competition in the interaction J_{ij}'s. The Lifshitz point corresponds to $p = 0$ when, as mentioned previously, there is an instability associated with the quadratic terms q_α^2 (momenta corresponding to the spatial directions having frustration). Clearly $p > 0$ corresponds to the ferromagnetic phase and $p < 0$ corresponds to the helical phase with modulated wave-vector $q_\alpha = p/2$.

We shall renormalize the action $\mathcal{A}$, employing the ϵ-expansion around the upper critical dimension which in the present case is $d_u = 3 + m/2$. As in the classical case [9], here also two length scale renormalisation factors, denoted by a and b are involved corresponding to q_α's and q_β's respectively.

The renormalisation group equations for the parameter r and u, upto the first loop, are given as

$$\begin{aligned} r' &= b^2\left[r - 4(n+2)uA(r) + ...\right] \\ u' &= \zeta^4 a^{-3m} b^{-3(d-m)} b^{-3z}\left[u - 4(n+8)u^2 C(r)\right] \\ &= b^{(3+m/2)-d}\left[u - 4(n+8)u^2 C(r) + ...\right], \qquad (16)\end{aligned}$$

where the field renormalisation parameter $\zeta^2 = a^m b^{d-m} a^4 b^z = a^m b^{d-m} b^{2+z}$. From the renormalisation of u, it is clear that for $d > 4 + m/2 - z$, u is irrelevant and the model has essentially the Gaussian critical behaviour with exponents $\nu_{l4} = 1/4$, $\nu_{l2} = 2\nu_{l2} = 1/2$ and $\eta_{l2} = \eta_{l4} = 0$. Hence, the upper critical dimension in the anisotropic case $d_u = 4 + m/2 - z = 3 + m/2$. Defining $\epsilon = 4 + m/2 - (d + z) = 3 + m/2 - d$, another fixed point which we call the quantum Heisenberg Lifshtiz point (which is stable below the upper critical dimension), is given by

$$p = 0;\quad u^* = \frac{\epsilon \ln b}{4(n+8)C(0)}\cdot \qquad (17)$$

Linearising around the quantum Heisenberg Lifshitz point we find the exponents to the $O(\epsilon)$ given as

$$\nu_{l2} = \frac{1}{2} + \frac{\epsilon}{4}\left(\frac{n+2}{n+8}\right)$$

$$\nu_{l4} = \frac{1}{4} + \frac{\epsilon}{8}\left(\frac{n+2}{n+8}\right), \qquad (18)$$

to first order in ϵ. Obviously, η_{l4} and η_{l2} are of the order of $O(\epsilon^2)$.

Similarly, one now proceeds to the isotropic case. Here, since only one set of momenta ($q_\alpha = (q_1, q_2, ..., q_d)$) are involved, we do only need one length rescaling factor a and consequently one comes across ν_{l4} and η_{l4} only. Under renormalisation group transformation the functions u and v in present case, assume the form with the choice of ζ given by $\zeta^2 = a^{d+z+4}$

$$\begin{aligned} v(q) &= \left[r + \omega^2 a^{4-2z} + q_\alpha^4 + ...\right] \\ u'(q,\omega) &= a^{8-(d+z)}\left[u - 4(n+8)u^2 C(r) + ...\right]. \qquad (19)\end{aligned}$$

Clearly in the present case, we can readily see that the action invariance demands $z = 2$ so that as in the anisotropic case here also quantum effect is marginal upto the first order ϵ-expansion. From the renormalisation equation of u, one finds that here the upper critical dimension is 6 in contrast to the classical isotropic Lifshitz point where upper critical dimension is 8. This is interesting since in spite of the translational invariance and periodic nature of frustration the nature of the action demands the dimensional shift in case of the quantum isotropic Lifshitz point is 2 rather than unity [11]. The single loop integral which determines the "mass" renormalisation in the isotropic case is written as

$$I = \int \frac{d^d q}{(2\pi)^d} \frac{d\omega}{2\pi} \frac{1}{q^4 + \omega^2 + r} \ ,$$
$$\sim \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^4 + r)^{1/2}} \ . \tag{20}$$

It scales as $r^{(d-2)/4}$, while in the corresponding classical case, I scales as $\int (d^d q/(2\pi)^d (1/(r+q^4))$ and hence $I \sim r^{(d-4)/4}$. Clearly the quantum situation in d dimensions correspond to the classical situation in $(d+2)$ dimensions. Since 6 is the upper critical dimension in the isotropic case, the fixed point value of u is now $O(6-d)$ in the lowest order and straightforward application of the standard renormalisation group technique yields for $\epsilon = 6 - d$ [11]

$$\nu_{l4} = \frac{1}{4} + \frac{\epsilon}{16}\left(\frac{n+2}{n+8}\right), \tag{21}$$

which perfectly matches the ν obtained by Hornreich *et al.* [9] for the classical case when $\epsilon = 8 - d$. The lowest order η is found from a two loop calculation and in accordance with our expectation, matches with the value of η in $O(\epsilon^2)$ [9]:

$$\eta = -\frac{3}{20}\frac{n+2}{(n+8)^2}\epsilon^2. \tag{22}$$

To evaluate the exponents to the second order in ϵ in anisotropic case, we need to perform a double ϵ expansion around d_0 and m_0 [9,10]. In the isotropic case, it is checked upto the second order in ϵ that exponents for quantum phase transition exactly match with the exponents associated with the classical Lifshitz point.

4 Scaling behaviour near the Lifshitz point

We shall now determine the scaling relations in the neighbourhood of a general (d,m) quantum Lifshitz point. The ground state energy density (with $T=0$) for action $\mathcal{A}$ (2)

$$E_d = E_d\left(r, q_\alpha, q_\beta, \omega\right), \tag{23}$$

is a function of r, q's and ω, where r measures the interval from the Lifshitz point. Under the renormalisation group transformation as defined previously by a factor a or b the ground state energy density defined above scales as

$$E_d \sim b^{-(d-m)} a^{-m} b^{-z} g\left(r b^{1/\nu_{l2}}, q_\beta b, q_\alpha a, \omega b^z\right),$$

where g is the scaling function. Choosing, $r b^{1/\nu_{l2}} = 1$ or equivalently $r a^{1/\nu_{l4}} = 1$ we obtain

$$E_d \sim |r|^{(d-m+z)\nu_{l2}} |r|^{m\nu_{l4}} g\left(q_\beta/|r|^{\nu_{l2}}, q_\alpha/|r|^{\nu_{l4}}, \omega/|r|^{\nu_{l2} z}\right).$$

Clearly the temporal correlation length ξ_τ diverges as r^{ν_τ} so that $\nu_\tau = \nu_{l2} z$. The free energy density is expected to scale in the form $|r|^{2-\alpha_l}$ so that we obtain a modified hyperscaling relation associated with the transition at a (d,m) quantum Lifshitz point given as

$$2 - \alpha_l = (d - m + z)\nu_{l2} + m\nu_{l4}, \tag{24}$$

with $z = 1$. In the isotropic case, similar arguments will lead to the modified hyperscaling relation

$$2 - \alpha_l = \nu_{l4}(d + z'), \tag{25}$$

with $z' = 2$ as mentioned previously. Again in this case, the temporal correlation length exponent $\nu_\tau = z\nu_{l4}$.

We shall now consider the scaling of the dynamic susceptibility under the previous renormalisation group transformation given as

$$\chi \propto \zeta^2 a^{-m} b^{-(d-m)} \tag{26}$$

where ζ, as defined previously, is given by

$$\zeta^2 = a^m b^{d-m} a^{4-\eta_{l4}} = a^m b^{d-m} b^{2-\eta_{l2}}.$$

With the above choice of a and b one finds for the scaling relation of the correlation function as

$$\chi \propto |r|^{-(2-\eta_{l2})\nu_{l2}}.$$

As χ diverges as $|r|^{-\gamma}$, one readily comes across the scaling relation

$$\gamma_l = (2 - \eta_{l2})\nu_{l2}, \tag{27a}$$

or equivalently

$$\gamma_l = (4 - \eta_{l4})\nu_{l4}. \tag{27b}$$

In the isotropic case, only ν_{l4}'s are involved so one finds the scaling relations of the form

$$2 - \alpha_l = \nu_{l4}(d + z'), \tag{28}$$
$$\gamma_l = (4 - \alpha_{l4})\nu_{l4},$$

with $z' = 2z = 2$ in the present case.

5 Exponents in the spherical limit and the Fisher renormalisation

We shall study action $\mathcal{A}$ (Eq. (15)) in the spherical limit $(n \to \infty)$ with u in action A is of the order of $1/n$. Following Ma [15], we shall only consider the "bubble diagrams" which contribute in the $n \to \infty$ limit. The self-energy correction just above the quantum critical point in the $n \to \infty$ limit, in the anisotropic case, is given as

$$\Sigma_a(r) - \Sigma_a(0) =$$
$$\int \frac{d\omega}{2\pi} \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{\omega^2 + (q_\alpha^2)^2 + q_\beta^2 + r} - \frac{1}{\omega^2 + (q_\alpha^2)^2 + q_\beta^2} \right)$$
$$= \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{((q_\alpha^2)^2 + q_\beta^2 + r)^{1/2}} - \frac{1}{((q_\alpha^2)^2 + q_\beta^2)^{1/2}} \right)$$
$$\sim r^{(d-1)/2 - m/4}. \tag{29}$$

Since, near the quantum critical point $\Sigma_a(r) - \Sigma_a(0) \sim (r)^{1/\gamma_l}$,hence we find for $1 + m/2 < d < 3 + m/2$, $1/\gamma_l = (d-1)/2 - m/2$, so that $\gamma_l = 2(d - 1 - m/2)^{-1}$. Since η_{l2} and η_{l4} are zero in the spherical limit, one can find using the scaling relations (27), $2\nu_{l4} = \nu_{l2} = (d-1-m/2)^{-1}$. Similar calculations in the isotropic case yields $\gamma_l = 2(d/2-1)^{-1}$; $\nu_{l4} = \frac{1}{2}(d/2-1)^{-1}$.

To evaluate the exponent α_l, we note that lowest order contribution to the specific heat near the critical point ($q_\alpha \to 0$, $q_\beta \to 0$ and $\omega \to 0$) is written as

$$C = n\prod(r,0)\left(1 + nu\prod(r,0)\right)^{-1} \tag{30}$$

where the loop contribution in the anisotropic case is written as

$$\prod(r,0,0) = \int \frac{d^d p}{(2\pi)^d}\int \frac{d\omega}{2\pi}\frac{1}{(\omega^2 + (q_\alpha^2)^2 + q_\beta^2 + r)^2} \tag{31}$$

$$\sim \int \frac{d^d p}{(2\pi)^d}\frac{1}{((q_\alpha^2)^2 + q_\beta^2 + r)^{3/2}}$$

$$\sim \int \frac{d^m q_\alpha}{(2\pi)^d}\frac{1}{((q_\alpha^2)^2 + r)^{(3+m-d)/2}}$$

$$\sim r^{(d-3-m/2)/2} \tag{32}$$

so that for $1+m/2 < d < 3+m/2$, the specific heat exponent α_l is written as $\alpha_l = (d-3-m/2)/(d-1-m/2)$. Similarly for the isotropic case, one has $\alpha_l = (d/2-3)/(d/2-1)$. With the above value of α_l, one can now verify the hyperscaling relations, which is satisfied in the isotropic case with $z' = 2z = 2$.

In the case of classical n-vector Hamiltonian the exponents associated with the thermal phase transitions in the spherical limit, can be reduced to the "Fisher renormalised" Gaussian exponents [16,17]. It can be shown that this equivalence between the Gaussian and the spherical model exponents can be extended even in the case of zero-temperature quantum transitions (see Appendix A). Hence, the exponents in the spherical limit are related to the exponents in the Gaussian limit are related by

$$\alpha_s = -\frac{\alpha_g}{1-\alpha_g}$$

$$\gamma_s = \frac{\gamma_g}{1-\alpha_g}$$

$$\nu_s = \frac{\nu_g}{1-\alpha_g} \tag{33}$$

where the subscript g (s) denote the Gaussian (spherical) exponents. In the Gaussian limit, the exponents at the Lifshitz point are simply $\nu_{l2}^g = 1/2$, $\nu_{l4}^g = 1/4$ and $\gamma_l^g = 1$. Using the hyperscaling relation we find $\alpha_l^g = 1 - d/2 + 1/2 + m/4$. The Fisher renormalised exponents (exponents in the spherical limit) obtained from the equations (33) are $\gamma_l^s = 2(d-1-m/2)^{-1}$, $\alpha_l^s = (d-3-m/2)/(d-1-m/2)$ which match perfectly with the exponents obtained using the standard renormalisation group technique. It may be mentioned that the dynamic exponent remains unchanged under this renormalisation: $z_s = z_g$.

6 Conclusion

We have studied the quantum rotor system in the presence of regular frustration in the interaction. The phase diagram in the vicinity of the quantum Lifshitz point and also the condition for the existence of the helical phase has been studied in the spherical ($n \to \infty$) limit. In the finite n case, the standard renormalisation group flow equations (one loop approximation) have been employed to extract the critical exponents associated with the quantum Lifshitz point. The dynamical exponent z of the quantum system has been found to be 2 in the case of isotropic Lifshitz point while in the anisotropic case z is 1. We have also derived the modified hyperscaling relation starting from a generic ground state energy density associated with the quantum phase transitions. The exponents are also derived in the spherical limit and it has been shown explicitly that the even in the case of quantum phase transitions the exponents in the spherical limit are related to the exponents in the Gaussian limit by "Fisher renormalisation".

Appendix A: Fisher renormalisation of quantum spherical models

In this section, we shall indicate how the relationship between the exponents associated with the quantum phase transition in the spherical limit to those in the corresponding Gaussian model is obtained through "Fisher renormalisation". Following Fisher [17], the critical exponents of a system with annealed coupling to analytically constrained variables can be expressed by Fisher renormalisation of the critical exponents of the unconstrained or pure systems. In the following, we shall show that even in zero temperature quantum transition, the spherical model can be expressed as a Gaussian model with annealed (analytic) constraints [16].

We shall again start here with the partition function of the n-component soft rotor as given previously in equation (14). We now employ the Hubbard-Stratonowitch transformations to simplify the term

$$\exp\left(-\frac{u}{4}\int_0^\beta \{\sum_{\alpha=1}^n x_i^2\}^2\right) = (\pi u)^{-1/2}$$

$$\times \int \prod_i d\psi_i \exp\Bigg(-\frac{1}{u}\int_0^\beta \psi_i^2(\tau)d\tau$$

$$+ i\int_0^\beta d\tau \psi_i(\tau)\sum_{\alpha=1}^n x_{i\alpha}^2(\tau)\Bigg); \tag{A.1}$$

so that

$$Z=(\pi u)^{-N/2}\int\prod_i d\psi_i \exp\Big(-\frac{1}{u}\sum_{\alpha=1}^{n}\int_0^\beta d\tau\psi_i^2(\tau)d\tau$$

$$+i\int_0^\beta d\tau\psi_i(\tau)\sum_{\alpha=1}^{n}x_{i\alpha}^2(\tau)\Big)$$

$$-\int_0^\beta\sum_\alpha\Big(\sum_i\frac{1}{2g}[(\partial_\tau x_{i\alpha}(\tau))^2+rx_{i\alpha}^2)$$

$$+\sum_{ij}J_{ij}x_{i\alpha}(\tau)x_{j,\alpha}(\tau)\Big),$$

$$=\int_{-\infty}^{+\infty}\prod_i d\psi_i(\tau)\exp\left(\int_0^\beta d\tau\Big(-\frac{n}{u}\sum_i\psi_i^2(\tau)-n\phi_G(\psi)\Big)\right)$$

where the Gaussian free energy functional $\phi_G(\psi)$ is given by

$$e^{-n\phi_G(\psi)}=$$

$$\int\prod_i\prod_\alpha d\tau dx_{i\alpha}(\tau)\exp\Big(\sum_{\alpha=1^n}(-\sum_{ij}J_{ij}x_{i\alpha}(\tau)x_{j\alpha}(\tau)$$

$$+\sum_i(r+i\psi_i)x_{i\alpha}(\tau)^2+(1/2g)\sum_i(\partial_\tau x_{i\alpha}(\tau))^2)\Big).$$

For the large n, the saddle point condition is given by

$$\psi_i(\tau)=\frac{\partial\phi_G(\psi_i(\tau))}{\partial\psi_i(\tau)}, \qquad (A.2)$$

which may be taken to be the analytic constraint for the annealed coupling parameters ψ. One possible solution of the above saddle point equation is

$$\psi_i(\tau)=\psi,$$

for all i and τ. Following the same arguments as given in Emery [16] one now finds that the Fisher renormalisation is valid even in the quantum transitions and the quantum critical exponents in the spherical limit α_s, β_s and ν_s are related to the corresponding Gaussian exponents α_G, β_G and ν_G through the Fisher renormalisation (33).

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